MATH1520 University Mathematics for Applications **Spring 2021**

Chapter 11: Ordinary Differential Equations

Learning Objectives:
(1) Solve first-order linear differential equations and initial value problems.
(2) Explore analysis with applications to dilution models.

1 Ordinary Differential Equations

Definition 1.1. An ordinary differential equation (ODE) is an equation involving one or more derivatives of an unknown function $y(x)$ of 1-variable. A differential equation for a multi-variable function is called a "partial differential equation" (PDE).

The order of an ordinary differential equation is the order of the highest derivative that it contains.

Example 1.1. f_i, g are functions of x

DIFFERENTIAL EQUATION	ORDER
$\frac{dy}{dx} = 4x$	1
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y \cancel{-1}) = e^t + t$	3
$y' + y = 2x^2$	1

$f_0 = 1,\ f_1 = 0$, $g = 4x$

$f_1(x)\, y' + f_0 y = g$

linear ODE

$f_3 = 1,\ f_2 = 0,\ f_1 = -t,\ f_0 = t$, $g = e^t + t$

$f_3(t)\, y''' + f_2(t)\, y'' + f_1(t)\, y' + f_0 \cdot y = g$

$f_1 = 1,\ f_0 = 1,\ g = 2x^2$

Example 1.2.

1. $yy'' + e^y = x^2 \ln y'$ is a second order ODE. ← nonlinear
2. $f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x)$, $f_2(x) \neq 0$. This is a second order *linear* ODE in the function $y(x)$. $g(x)$ is called the *inhomogeneous term*; the left hand side of the equation is called the *homogeneous part* of the this linear ODE; $f_2(x)y" + f_1(x)y' + f_0(x)y = 0$ is called the associated homogeneous linear ODE of the linear ODE given above. A linear ODE with inhomogeous term 0 is called a *homogeneous* linear ODE.
3. The ODE in 1. is non-linear. The second ODE in Example 1.1 is linear with inhomogeneous term e^t.

homogeneous part. inhomogeneous term

Remark. $\sum_{i=1}^{n} a_i x_i = b$ where a_i, b are constants ("coefficients") is said to be a linear equation in the variables $x_1, \ldots, x_n$. b is called the inhomogeneous term, and the equation is said to be homogeneous when $b = 0$. For differential equations, functions of x play the roles of "coefficients" $a_1, \ldots, a_n$, b, and $y^{(i)}$, $i = 0, 1, \ldots$ play the roles of "variables".

Definition 1.2. A function $y = y(x)$ is a **solution** of an ordinary differential equation on an open interval if the equation is satisfied identically on the interval when y and its derivatives are substituted into the equation.

Remark. The solution might not exist; it might not be unique. homogeneous linear ODE

Example 1.3. $y(x) = e^{2x}$ is a solution to the ODE $y'' - 4y' + 4y = 0$. $y(x) = 4e^{2x}$ is another solution.

$\rightarrow (e^{2x})'' - 4(e^{2x})' + 4e^{2x} = 4e^{2x} - 4\cdot 2e^{2x} + 4e^{2x} = 0$

Example 1.4. Find the solution of $\frac{d}{dx}y = 4x$, or equivalently, $y'(x) = x$. 4

Solution. Integrate both sides: $y(x) = \int 4x\,dx = 2x^2 + C$, where C is an arbitrary constant.

Then, $y = 2x^2 + C$, $\quad C \in \mathbb{R}$ is called general solution of $y'(x) = 4x$.

Choose any C, e.g. $C = 5$, we get a particular solution $y = 2x^2 + 5$. ■

For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary x-value x_0, say $y(x_0) = y_0$. This is called an initial condition, and the problem of solving a first-order equation subject to an initial condition is called a **first-order initial-value problem**.

Example 1.5.

$$\begin{cases} y'(x) = 4x \\ y(5) = 20 \end{cases}$$

is an initial value problem.

$y(5) = 2\cdot 25 + C = 20.$

General solution $y = 2x^2 + C$ should satisfy the initial condition $y(5) = 20$, i.e.

$$20 = 2(5)^2 + C \quad \Rightarrow \quad C = -30.$$

So, the unique solution to the initial value problem is $y = 2x^2 - 30$.

Solving a general ODE is typically very difficult, and there is no general algorithm for doing so. We shall discuss only some particularly simple cases.

In general, for a n-th order ODE typically n "initial conditions" are required to get a unique solution.

Initial Conditions: typical $y(x_0) = a_0$ $y'(x_0) = a_1$ $+ \cdots$ $y^{(n-1)}(x_0) = a_{n-1}$

2 Separation of Variables for first order ODE

Definition 2.1 (Separable Equation)**.**

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is called a separable equation.

For those separable differential equations, we can formally rewrite them in the form ("separation of variables"–each side involve one single variable)

$$\text{“}h(y)\,dy = g(x)\,dx\text{”} \tag{1}$$

Integrate both sides with respect to x and y respectively, we have

$$\int h(y)\,dy = \int g(x)\,dx \tag{2}$$

or, equivalently

$$H(y) = G(x) + C \tag{3}$$

where $H(x)$, $G(x)$ denote antiderivatives of $h(x)$ and $g(x)$ respectively, and C denotes a constant.

Example 2.1. Solve

$$(1)\quad \frac{dy}{dx} = \frac{2x}{y^2} \qquad\qquad (2)\quad \begin{cases} \dfrac{dy}{dx} = \dfrac{2x}{y^2}, \\ y(0) = 1. \end{cases}$$

IVP

Solution. (1) Separating variables and integrating yields

$$\text{“}\, y^2 dy = 2x dx \,\text{”}$$

$$\int y^2\,dy = \int 2x dx$$

or

$$\frac{1}{3}y^3 = x^2 + C$$

or, equivalently

$$y = \sqrt[3]{3(x^2 + C)}$$

□.

$$y(0) = \sqrt[3]{3\cdot C} = 1 \quad \rightsquigarrow \quad C = \frac{1}{3}.$$

(2) The initial condition $y(0) = 1$ requires that $y = 1$ when $x = 0$. Substituting these values into our solution yields $C = \frac{1}{3}$ (verify). Thus, a solution to the initial-value problem is

$$y = \sqrt[3]{3x^2 + 1}.$$

■

Example 2.2. Solve

$$\frac{dy}{dx} = -4xy^3$$

"$\frac{dy}{y^3} = -4x\,dx$"

Solution. (1) For $y \neq 0$, we can write the differential equation as

$$\frac{1}{y^3}\frac{dy}{dx} = -4x$$

Separating variables and integrating yields

$$\frac{1}{y^3}dy = -4xdx$$

$$\int \frac{1}{y^3}dy = \int -4xdx$$

or

$$-\frac{1}{2y^2} = -2x^2 + C$$

$\Rightarrow -\frac{1}{2} \cdot \frac{1}{-2x^2+C} = y^2$

or, equivalently

$$y^2 = \frac{1}{4x^2 - 2C}$$

$y = \pm \frac{1}{\sqrt{4x^2-2C}}$

(2) Constant function $y = 0$ also satisfies the differential equation, since

$$0' = -4x \cdot (0)^3$$

Therefore, the solution is $y^2 = \frac{1}{4x^2-2C}$ or $y = 0$.

■

Remark. For $y' = g(x)h(y)$, divide both sides by $h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$.

Do not miss the particular constant solution $y = a$ that makes $h(a) = 0$.

Example 2.3. Solve $y' = 3x^2y$. $= \frac{dy}{dx}$

(if $y = 0$. the ODE also holds so $y = 0$ is a solution)

$\int 3x^2 dx = \int \frac{dy}{y}$ if. $y \neq 0$

Solution. (1) For $y \neq 0$, it can be written as

$$\frac{dy}{y} = 3x^2\,dx$$

so

$$\begin{aligned}
\int \frac{dy}{y} &= \int 3x^2\,dx \\
\ln|y| &= x^3 + C_1 \\
|y| &= e^{x^3} \cdot e^{C_1}, \qquad C_1 \in \mathbb{R} \\
y &= \pm e^{x^3} \cdot e^{C_1}, \qquad C_1 \in \mathbb{R} \\
y &= C_2 e^{x^3}, \qquad C_2 \neq 0
\end{aligned}$$

arbitrary positive number

(2) Check: $y = 0$ is also a solution.

Therefore, the general solution is

$$y = Ce^{x^3}, \qquad C \in \mathbb{R}$$

■

Example 2.4. Find a curve $y = y(x)$ on the $x - y$ plane that passes through $(0, 2)$ and whose tangent line at a point (x, y) has slope $2x^3/y^2$.

Solution. Since the slope of the tangent line is dy/dx, we have

$$\frac{dy}{dx} = \frac{2x^3}{y^2}$$

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x^3}{y^2} \leftarrow \text{slope of the tangent line at } (x,y) \\ y(0) = 2 \end{cases}$$

IVP

which is separable and can be written as

$$y^2 dy = 2x^3 dx$$

so

$$\int y^2 dy = \int 2x^3 dx \quad \text{or} \quad \frac{1}{3}y^3 = \frac{1}{2}x^4 + C$$

$\leftarrow$ plug in $x = 0$, $y = 2$

It follows from the initial condition that $y = 2$ if $x = 0$. Substituting these values into the last equation yields $C = \frac{8}{3}$ (verify), so the equation of the desired curve is

$$\boxed{\frac{1}{3}y^3 = \frac{1}{2}x^4 + \frac{8}{3}.}$$

$\frac{1}{3} \cdot 2^3 = \frac{1}{2} \cdot 0^4 + C$

$\frac{8}{3} = C$

■

3 First-Order Linear Differential Equations

Recall: A 1st order linear ODE has the general form $a(x)y' + b(x)y = c(x)$, where $a(x) \neq 0$. We can always divide the whole equation by $a(x)$ and consider equivalently the equation $y' + \frac{b}{a}y = \frac{c}{a}$ wherever $a(x) \neq 0$. So we may restrict to equations of the form

$$\frac{dy}{dx} + p(x)\, y = q(x). \tag{4}$$

(1) If $q(x) = 0$ (homogeneous case),

$$\frac{dy}{dx} + p(x)y = 0, \qquad \text{separable equation!}$$

$\frac{dy}{dx} = -p(x)\, y$

$\int \frac{dy}{y} = -\int p(x)\, dx$ if $y \neq 0$

(2) For general $q(x)$, use integrating factors!

Idea: multiply the differential equation by a factor $\mu(x)$ $\neq 0$, then

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

a function of x times y $(\mu(x))$

Hope we can rewrite LHS in the form of $\frac{d}{dx}(\cdots)$, then the differential equation can be written as

$$\frac{d}{dx}(\overset{\mu y}{\cdots}) = \mu(x)q(x) \qquad \text{separable equation!}$$

$\rightsquigarrow (\cdots) = \int \mu(x)\, q(x)$

Check: $\boxed{\mu(x) = e^{\int p(x)\, dx}}$ works!

fundamental thm of calculus

$\frac{d\mu}{dx} = e^{\int p(x)\,dx} \cdot \frac{d}{dx}\int p(x)\,dx = e^{\int p(x)\,dx}\, p(x) = \mu p.$

chain rule.

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu\frac{dy}{dx} + \frac{d\mu}{dx}y && \text{(product rule)} \\ &= \mu\frac{dy}{dx} + \mu p(x)y && \text{(chain rule)} \\ &= \mu q && \text{(apply equation)} \end{aligned}$$

So, $\mu y = \int \mu q\, dx$ and

$$\boxed{y = \frac{1}{\mu}\int \mu q\, dx}$$

C

Remark. There are infinitely many choices for $\mu(x) = e^{\int p(x)\, dx}$ (it involves an indefinite integral). Just pick any one!

The Method of Integrating Factors

Step 1. Calculate the integrating factor

$$\mu = e^{\int p(x)dx}.$$

Since any μ will suffice, we can take the constant of integration to be zero in this step.

Step 2. Multiply both sides of (4) by μ and express the result as

$$\frac{d}{dx}(\mu y) = \mu q(x).$$

Step 3. Integrate both sides of the equation obtained in Step 2 and then solve for y. Be sure to include a constant of integration in this step.

Example 3.1. Solve the differential equation

$$\frac{dy}{dx} - y = e^{3x}.$$

$P(x) = -1$ $\quad$ $q(x) = e^{3x}$

Solution. We have a first-order linear equation with $p(x) = -1$ and $q(x) = e^{3x}$.

$$\mu = e^{\int p(x)dx} = e^{\int(-1)dx} = e^{-x}.$$

Next we multiply both sides of the given equation by μ to obtain

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x}$$

which we can rewrite as

$$\frac{d}{dx}[e^{-x}y] = e^{2x}.$$

So

$$e^{-x}y = \frac{1}{2}e^{2x} + C$$

Finally, solving for y yields the general solution

$$y = \frac{1}{2}e^{3x} + Ce^{x}.$$

$\blacksquare$

Exercise 3.1. Solve $y' + 2xy = 4x$.
Ans: $y = 2 + Ce^{-x^2}$.

$P(x) = 2x$ $\quad$ $\mu(x) = e^{\int p\,dx} = e^{x^2}.$

$\frac{d}{dx}(e^{x^2}y) = e^{x^2}y' + 2xe^{x^2}y = 4xe^{x^2}.$

$e^{x^2}y = \int 4xe^{x^2}dx$ $\quad$ let $u = x^2$. $du = 2x\,dx$

$= \int 2e^{u}du = 2e^{u} + C = 2e^{x^2} + C$

$y = x + Ce^{-x^2}$

Example 3.2. Solve the initial-value problem

$$x\frac{dy}{dx} - y = x, \quad y(1) = 2.$$

Solution. By dividing both sides by x, we have

$$\frac{dy}{dx} - \frac{1}{x}y = 1, \quad (x \neq 0) \tag{5}$$

$p(x) = \frac{-1}{x}$ $\quad q = 1.$

By the initial condition at $x = 1$, we restrict domain to $x > 0$. Then,

$\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{|x|}$

$$\mu = e^{\int p(x)\, dx} = e^{-\int \frac{1}{x}\, dx} = e^{-\ln|x|} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying both sides of Equation (5) by this integrating factor yields

pick, $\mu = \frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$$

or

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{1}{x}$$

Therefore, on the interval $(0, +\infty)$,

$$\frac{1}{x}y = \int \frac{1}{x}\, dx = \ln|x| + C$$

from which it follows that

where $x \neq 0$

$$y = x\ln|x| + Cx. \tag{6}$$

$y(1) = C = 2$

By $y(1) = 2$, we have $C = 2$ (verify). So the solution of the initial-value problem is

$$y = x\ln x + 2x, \quad x > 0.$$

y is defined only on $(-\infty, 0) \cup (0, \infty)$. $2 \in (0, \infty)$, so the initial condition only determines C over $(0, \infty)$ ■

Exercise 3.2. Solve the initial-value problem

$$x\frac{dy}{dx} - y = x, \quad y(-1) = 2.$$

$-1 \in (-\infty, 0)$ so the initial condition only determines y over $(-\infty, 0)$ over $(-\infty, 0)$, C can still be arbitrary

$y(-1) = (-1)\cdot 0 + C(-1) = 2 \rightsquigarrow C = -2$

$\rightarrow y(x) = x\ln(-x) - 2x$ over $(-\infty, 0)$

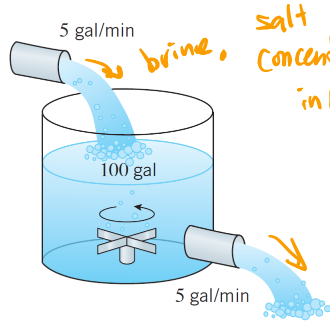
Ex. Find $y(x)$ satisfying $x\frac{dy}{dx} - y = x$, $y(-1) = 2$, $y(1) = 2$

Ans: $y(x) = \begin{cases} x\ln x + 2x & \text{where } x > 0 \\ x\ln(-x) - 2x & \text{when } x < 0 \end{cases}$

$= x\ln|x| + 2|x|$ when $x \neq 0$ ∎

4 Modeling with ODE

Example 4.1 (Mixing Problem). At time $t = 0$, a tank contains 4 lb of salt dissolved in 100 gal of water. Suppose that brine containing 2 lb/gallon of salt is pumped into the tank at a rate of 5 gal/min. At the same time, that the well-mixed solution is drained from the tank at the same rate. Find the amount of salt in the tank after 10 minutes.



brine, salt concentration 2lb/gal.
incoming rate 5 gal/min

outgoing rate 5 gal/min

Solution.

$$\begin{aligned} \text{Let}\quad y(t) &= \text{amount of salt (lb) at time } t. \\ y(0) &= 4 \text{ lb.} \\ \text{Aim:}\quad y(10) &= \ ? \end{aligned}$$

← units minutes

Key: How $y(t)$ changes? or, $\dfrac{dy}{dt} = ?$ lb/min.

We always have

$$\boxed{\frac{dy}{dt} = \text{rate in} - \text{rate out.}}$$

where rate in is the rate at which salt enters the tank and rate out is the rate at which salt leaves the tank.

By the formula: $\boxed{\text{mass} = \text{volume} \times \text{concentration}}$, we have

$$\text{rate in} = (2 \text{ lb/gal}) \cdot (5 \text{ gal/min}) = 10 \text{ lb/min.}$$

$$\text{rate out} = \left(\frac{y(t)}{100} \text{ lb/gal}\right) \cdot (5 \text{ gal/min}) = \frac{y(t)}{20} \text{ lb/min.}$$

Therefore, we have an initial first order linear ordinary differential equation

$$\rightarrow \begin{cases} \dfrac{dy}{dt} = 10 - \dfrac{y}{20} \quad \text{or} \quad \dfrac{dy}{dt} + \dfrac{y}{20} = 10 \\ y(0) = 4. \end{cases}$$

$\dfrac{dy}{dt} + py = q$

$p = \dfrac{1}{20}$, $q = 10$

The integrating factor for the differential equation is $\left(e^{t/20} y\right)' = e^{t/20} y' + \frac{1}{20} e^{t/20} y = 10 e^{t/20}$

$$\mu = e^{\int (1/20)dt} = e^{t/20}.$$

If we multiply the differential equation through by μ, then we obtain

$$\frac{d}{dt}(e^{t/20}y) = 10e^{t/20}$$

$$e^{t/20}y = \int 10e^{t/20}dt = 200e^{t/20} + C$$

$$y(t) = 200 + Ce^{-t/20}.$$

$y(0) = 200 + C = 4$

Substituting $t = 0$ and $y = 4$ into $y(t)$ and solving for C yields $C = -196$, so

$$y(t) = 200 - 196e^{-t/20}.$$

At time $t = 10$, the amount of salt in the tank is

$$y(10) = 200 - 196e^{-10/20} \approx 81.1 \text{ lb}.$$

■

Remark. After sufficiently long time, as $t \to +\infty$, $y(t) \to 200$ lb.

Example 4.2. Modelling a pandemic: (SIR model)

`https://www.youtube.com/watch?feature=share&v=Qrp40ck3WpI&app=desktop`

Note: the number of infected grows exponentially in the initial stages (no intervention).

Coronavirus Cases Live Updates:

`https://www.youtube.com/watch?feature=share&v=Qrp40ck3WpI&app=desktop`